Composition of Poisson Processes

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Abstract. *The main topic of this paper is the composition of independent Poisson processes of which we study the probability law (involving Bell polynomials), the governing equations, and its representation as a random sum. Some compositions of Poisson processes with the inverses of classical and fractional Poisson processes, and with the inverse of the fractional linear birth process, are examined and their pricipal features analysed.*

Keywords. *Poisson process, Iterated processes, Bell polynomials, Subordination, Fractional Poisson process, Fractional Yule–Furry process.*

1 Introduction

The subordination of processes, first introduced in [1], is a technique which permits us to introduce in various stochastic models, further randomness. The so-called Iterated Brownian motion, for instance, is a process which was extensively studied in the past years (see for example [2]).

Recently, the study of subordinated processes has been extended to point processes. It has been proved that the one-dimensional distributions of some subordinated point processes satisfy fractional difference-differential equations [3, 4]. This suggests us to study different kinds of subordinated point processes. In particular, in this paper, we study in details the iterated Poisson process as well as some other subordinated Poisson processes.

In Section 2, the iterated Poisson process is presented and analysed. Let $N_1(t)$ and $N_2(t)$, $t > 0$, be two independent homogeneous Poisson processes with rates $\lambda > 0$ and $\beta > 0$, respectively. The iterated Posson process is defined as $\hat{N}(t) = N_1(N_2(t))$, $t > 0$. The state probabilities $\hat{p}_k(t) = \Pr\{\hat{N}(t) = k\}$, $k \geq 0$, have the form

$$\hat{p}_k(t) = \frac{\lambda^k}{k!} e^{-\beta t(1-e^{-\lambda})} \mathfrak{B}_k\left(\beta t e^{-\lambda}\right), \tag{1}$$

with $k \geq 0$, $t > 0$, and where $\mathfrak{B}_k(x)$ is the nth order Bell polynomials.

By considering that the probability generating function of the process $\hat{N}(t) = N_1(N_2(t))$ turns out to be

$$\hat{G}(u,t) = e^{\beta t\left(e^{\lambda(u-1)}-1\right)}, \qquad t > 0,\ |u| \leq 1,$$

we establish the useful representation:

$$N_1(N_2(t)) \overset{\mathrm{d}}{=} X_1 + \cdots + X_{N_2(t)}, \tag{2}$$

where the X_j are i.i.d. Poisson-distributed random variables with parameter λ. The iterated Poisson process is thus, a compound Poisson process with Poisson components.

The difference-differential equations solved by the state probabilities is also derived and reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_k(t) = -\beta\hat{p}_k(t) + \beta e^{-\lambda} \sum_{m=0}^{k} \frac{\lambda^m}{m!} \hat{p}_{k-m}(t). \tag{3}$$

In Section 3 a Poisson process $N_1(t)$, $t > 0$, is subordinated to the inverse process of a second Poisson process $N_2(t)$ (Erlang process), independent of $N_1(t)$, i.e.

$$N_1(\tau_k), \qquad k \geq 0, \tag{4}$$

where

$$\tau_k = \inf(t\colon N_2(t) = k), \qquad k \geq 0, \tag{5}$$

and the explicit form of the state probabilities is given and discussed.

The last two sections are devoted to the study of Poisson processes subordinated to inverse processes of the fractional Poisson and fractional Yule process.

2 First results on subordinated Poisson processes

Consider two independent homogeneous Poisson processes, say $N_1(t)$, $t > 0$, with rate $\lambda > 0$, and $N_2(t)$, $t > 0$, with rate $\beta > 0$. We are interested in the subordinated process $\hat{N}(t) = N_1(N_2(t))$, $t > 0$. The subordination (see [1]) permits us to introduce in the system further randomness, thus allowing to model phenomena which exhibit either a speeded up or a slowed down behaviour. In particular, the type of subordination considered in this section is a little different. Poisson processes have a countable state space, therefore the inner Poisson process $N_2(t)$, $t>0$, in addition to randomize the time, operates a sampling of the external process $N_1(t)$, $t>0$. Note that this sampling allows the subordinated process to have jumps of arbitrary positive size. In addition, $\hat{N}(t)$, $t>0$, can be regarded as a Poisson process $N_1(t)$, $t>0$, running on the sample paths of the independent Poisson process $N_2(t)$, $t>0$.

The state probabilities $\hat{p}_k(t) = \Pr\{N_1(N_2(t)) = k\}$, $k \geq 0$, $t>0$, can be determined as follows.

$$\begin{aligned}\hat{p}_k(t) &= \sum_{r=0}^{\infty} \frac{e^{-\lambda r}(\lambda r)^k}{k!} \cdot \frac{e^{-\beta t}(\beta t)^r}{r!} \qquad (6)\\ &= \frac{\lambda^k}{k!} e^{-\beta t} \sum_{r=0}^{\infty} \frac{e^{-\lambda r} r^k (\beta t)^r}{r!}.\end{aligned}$$

By considering that

$$\mathfrak{B}_k(x) = e^{-x} \sum_{r=0}^{\infty} \frac{r^k x^r}{r!}, \qquad (7)$$

is the nth order Bell polynomial (for a review of Bell polynomials, the reader can consult for example [5]), we obtain that

$$\hat{p}_k(t) = \frac{\lambda^k}{k!} e^{-\beta t(1-e^{-\lambda})} \mathfrak{B}_k\left(\beta t e^{-\lambda}\right), \qquad k \geq 0,\ t>0. \qquad (8)$$

The following alternative form of the state probabilities is immediately calculated:

$$\begin{aligned}&\hat{p}_k(t) \qquad (9)\\ &= \frac{\lambda^k}{k!} e^{-\beta t} \sum_{r=0}^{\infty} \frac{e^{-\lambda r} r^k (\beta t)^r}{r!}\\ &= \Pr\{N_1(t) = k\} e^{\lambda t} \sum_{r=0}^{\infty} e^{-\lambda r} \left(\frac{r}{t}\right)^k \Pr\{N_2(t) = r\}.\end{aligned}$$

In Figure 1, we can appreciate the plots of the first four state probabilities for $\hat{N}(t)$, $t>0$, with $\lambda=1$ and $\beta=1$. By recalling that the exponential generating function for Bell polynomials is

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} \mathfrak{B}_k(x) = e^{x(e^z - 1)}, \qquad (10)$$

it is immediate to check that $\sum_{k=0}^{\infty} \hat{p}_k(t) = 1$. The mean value is directly calculated as follows.

$$\begin{aligned}\mathbb{E}\big[N_1(N_2(t))\big] &= \sum_{k=0}^{\infty} k \sum_{r=0}^{\infty} \frac{e^{-\lambda r}(\lambda r)^k}{k!} \frac{e^{-\beta t}(\beta t)^r}{r!} \qquad (11)\\ &= \sum_{r=0}^{\infty} \frac{e^{-\beta t}(\beta t)^r}{r!} e^{-\lambda r} \sum_{k=0}^{\infty} k \frac{(\lambda r)^k}{k!}\\ &= \sum_{r=0}^{\infty} \frac{e^{-\beta t}(\beta t)^r}{r!} e^{-\lambda r} \lambda r \sum_{k=0}^{\infty} \frac{(\lambda r)^k}{k!}\\ &= e^{-\beta t} \sum_{r=0}^{\infty} \frac{\lambda r (\beta t)^r}{r!}\\ &= e^{-\beta t} \lambda \beta t \sum_{r=0}^{\infty} \frac{(\beta t)^r}{r!}\\ &= \lambda \beta t, \qquad t>0.\end{aligned}$$

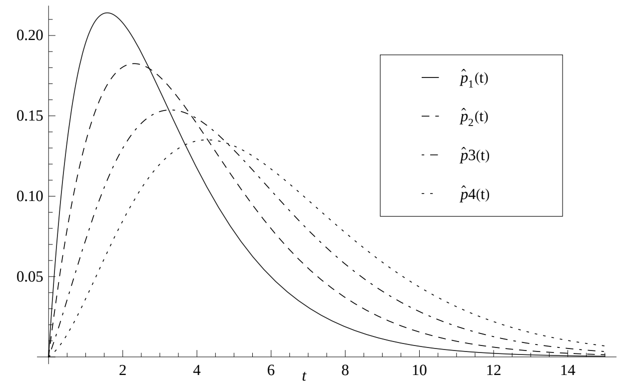

Figure 1: Graphs of the first four state probabilities for the Bell process. The parameters here have the values $\lambda = 1$, $\beta = 1$.

To determine the variance we first calculate the second order moment:

$$\begin{aligned}\mu_2 &= \sum_{k=0}^{\infty} k^2 \sum_{r=0}^{\infty} \frac{e^{-\lambda r}(\lambda r)^k}{k!} \frac{e^{-\beta t}(\beta t)^r}{r!} \qquad (12)\\ &= e^{-\beta t} \sum_{r=0}^{\infty} \frac{(\beta t)^r}{r!} e^{-\lambda r} \sum_{k=0}^{\infty} k^2 \frac{(\lambda r)^k}{k!}.\end{aligned}$$

Considering that

$$\begin{aligned}\sum_{k=0}^{\infty} k^2 \frac{(\lambda r)^k}{k!} &= \sum_{k=1}^{\infty} k \frac{(\lambda r)^k}{(k-1)!} = \sum_{k=0}^{\infty} (k+1) \frac{(\lambda r)^{k+1}}{k!} \qquad (13)\\ &= \sum_{k=1}^{\infty} k \frac{(\lambda r)^{k+1}}{k!} + \sum_{k=0}^{\infty} \frac{(\lambda r)^{k+1}}{k!}\\ &= \sum_{k=0}^{\infty} \frac{(\lambda r)^{k+2}}{k!} + \lambda r e^{\lambda r}\\ &= \lambda r e^{\lambda r}(\lambda r + 1),\end{aligned}$$

we have

$$\mu_2 = e^{-\beta t} \sum_{r=0}^{\infty} \frac{\lambda r(\lambda r + 1)(\beta t)^r}{r!} \qquad (14)$$

$$
\begin{aligned}
&= e^{-\beta t}\lambda \sum_{r=1}^{\infty} \frac{(\lambda r+1)(\beta t)^r}{(r-1)!} \\
&= e^{-\beta t}\lambda \sum_{r=0}^{\infty} \frac{(\lambda r+\lambda+1)(\beta t)^{r+1}}{r!} \\
&= e^{-\beta t}\lambda \left[(\lambda+1)\sum_{r=0}^{\infty}\frac{(\beta t)^{r+1}}{r!} + \lambda \sum_{r=1}^{\infty} r\frac{(\beta t)^{r+1}}{r!}\right] \\
&= \lambda(\lambda+1)\beta t + e^{-\beta t}\lambda^2 \sum_{r=0}^{\infty}\frac{(\beta t)^{r+2}}{r!} \\
&= \lambda(\lambda+1)\beta t + (\lambda\beta t)^2.
\end{aligned}
$$

Therefore, the variance of the process reads

$$
\mathbb{V}\mathrm{ar}\big[N_1(N_2(t))\big] = \lambda(\lambda+1)\beta t, \qquad t>0. \tag{15}
$$

The probability generating function

$$
\hat{G}(u,t) = \sum_{k=0}^{\infty} u^k \hat{N}(t) \tag{16}
$$

can be found in the following way:

$$
\begin{aligned}
\hat{G}(u,t) &= \sum_{k=0}^{\infty} u^k \sum_{r=0}^{\infty} \frac{e^{-\lambda r}(\lambda r)^k}{k!}\frac{e^{-\beta t}(\beta t)^r}{r!} \qquad (17)\\
&= e^{-\beta t\left(1-e^{-\lambda}\right)} \sum_{k=0}^{\infty} u^k \frac{\lambda^k}{k!}\mathfrak{B}\left(\beta t e^{-\lambda}\right) \\
&= e^{-\beta t\left(1-e^{-\lambda}\right)} e^{\beta t e^{-\lambda}\left(e^{u\lambda}-1\right)} \\
&= e^{\beta t\left(e^{\lambda(u-1)}-1\right)}, \qquad t>0,\ |u|\le 1.
\end{aligned}
$$

Result (17) implies that

$$
N_1(N_2(t)) \overset{\mathrm{d}}{=} X_1 + \dots + X_{N_2(t)}, \tag{18}
$$

where the random variables X_j are i.i.d. with Poisson distribution with parameter λ. In other words, the subordinated process $\hat{N}(t)$, $t>0$, is a compound Poisson process with Poisson components.

The difference-differential equations solved by the state probabilities $\hat{p}_k(t)$, $t>0$, is

$$
\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_k(t) = -\beta\hat{p}_k(t) + \beta e^{-\lambda}\sum_{m=0}^{k}\frac{\lambda^m}{m!}\hat{p}_{k-m}(t), \tag{19}
$$

and can be derived with the following steps.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_k(t) \qquad (20)\\
&= -\beta\hat{p}_k(t) + \beta\frac{\lambda^k}{k!}e^{-\beta t}\sum_{r=1}^{\infty} e^{-\lambda r}\frac{r^k(\beta t)^{r-1}}{(r-1)!} \\
&= -\beta\hat{p}_k(t) + \beta\frac{\lambda^k}{k!}e^{-\beta t}\sum_{r=0}^{\infty} e^{-\lambda(r+1)}\frac{(r+1)^k(\beta t)^r}{r!}
\end{aligned}
$$

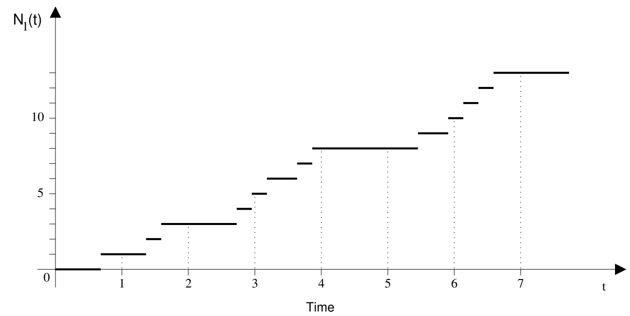

Figure 2: A possible path of the external Poisson process $N_1(t)$, $t>0$.

$$
\begin{aligned}
&= -\beta\hat{p}_k(t) + \beta e^{-\beta t}\frac{\lambda^k}{k!}e^{-\lambda} \\
&\quad\times \sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\sum_{r=0}^{\infty} e^{-\lambda r} r^m \frac{(\beta t)^r}{r!} \\
&= -\beta\hat{p}_k(t) + \beta\lambda^k e^{-\lambda}\sum_{m=0}^{k}\frac{\lambda^{-m}}{(k-m)!}\hat{p}_m(t) \\
&= -\beta\hat{p}_k(t) + \beta e^{-\lambda}\sum_{m=0}^{k}\frac{\lambda^m}{m!}\hat{p}_{k-m}(t).
\end{aligned}
$$

From result (19), the differential equation governing the probability generating function is easily derived and reads

$$
\frac{\partial}{\partial t}\hat{G}(u,t) = -\beta\hat{G}(u,t) + \beta e^{-\lambda}\hat{G}(u,t)e^{\lambda u}. \tag{21}
$$

With initial condition $\hat{G}(u,0)=1$ this yields result (17).

Remark 1 (A note on paths)**.** *It is interesting to describe the path behaviour of the subordinated process $\hat{N}(t) = N_1(N_2(t))$, $t>0$. In Figure 2, a possible path of the external Poisson process $N_1(t)$, $t>0$, is depicted. Since the internal process has discrete state space, it produces a discretisation of the time argument of the external process. Furthermore the process $N_1(N_2(t))$ admits arbitrarily high jumps. After the appearance of the first component, the process behaves as a pure birth process with a random number of offsprings.*

3 Poisson process subordinated to the inverse of a second independent Poisson process

We now consider the inverse process (hitting time) of the Poisson process $N_2(t)$, $t>0$:

$$
\tau_k = \inf(t\colon N_2(t)=k), \qquad k\ge 0. \tag{22}
$$

The process τ_k, $k\ge 0$, is a discrete time process with continuous state-space. It is a non-decreasing process with

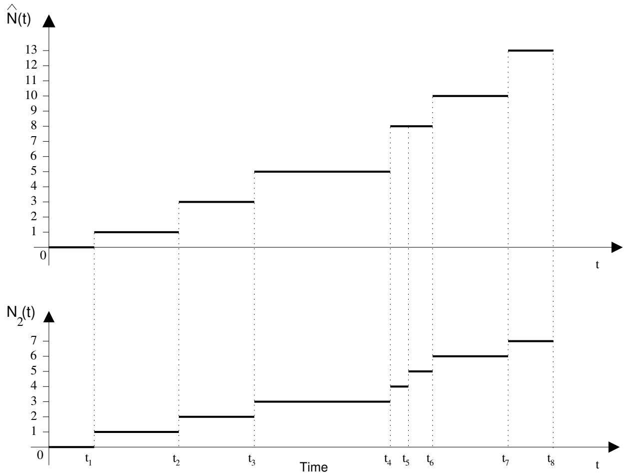

Figure 3: Construction of the path of the subordinated process $\hat{N}(t) = N_1(N_2(t))$, $t > 0$, starting from the path of the internal process $N_2(t)$, $t > 0$.

scattered jumps. In practice it can also be considered as the waiting time until the kth event:

$$\begin{aligned} \Pr(\tau_k \le t) &= \Pr(N_2(t) \ge k) \qquad (23) \\ &= \sum_{j=k}^{\infty} \frac{e^{-\beta t}(\beta t)^j}{j!} \\ &= 1 - \sum_{j=0}^{k-1} \frac{e^{-\beta t}(\beta t)^j}{j!}. \end{aligned}$$

Clearly τ_k, $k \ge 1$, is Erlang-distributed:

$$\Pr(\tau_k \in \mathrm{d}t)/\mathrm{d}t = \frac{\beta(\beta t)^{k-1}}{(k-1)!} e^{-\beta t} \mathrm{d}t. \qquad (24)$$

Our aim is to study the subordinated process $\tilde{N}(k) = N_1(\tau_k)$, $k \ge 0$.

$$\begin{aligned} &\Pr(N_1(\tau_k) = r) \qquad (25) \\ &= \int_0^{\infty} \frac{e^{-\lambda s}(\lambda s)^r}{r!} \frac{\beta(\beta s)^{k-1}}{(k-1)!} e^{-\beta s} \mathrm{d}s \\ &= \frac{\lambda^r \beta^k}{r!(k-1)!} \int_0^{\infty} e^{-(\lambda+\beta)s} s^{r+k-1} \mathrm{d}s \\ &= \frac{\lambda^r \beta^k}{r!(k-1)!} \int_0^{\infty} e^{-y} y^{r+k-1} \frac{1}{(\lambda+\beta)^{r+k}} \mathrm{d}y \\ &= \frac{\lambda^r \beta^k \Gamma(r+k)}{r!(k-1)!(\lambda+\beta)^{r+k}} \\ &= \frac{(r+k-1)!}{r!(k-1)!} \frac{\lambda^r \beta^k}{(\lambda+\beta)^{r+k}} \\ &= \binom{r+k-1}{r} \frac{\lambda^r \beta^k}{(\lambda+\beta)^{r+k}} \\ &= \binom{r+k-1}{r} \left(\frac{\lambda}{\lambda+\beta}\right)^r \left(1 - \frac{\lambda}{\lambda+\beta}\right)^k, \end{aligned}$$

$(r \ge 0,\ k \ge 0)$ which is a negative binomial distribution with parameter $\lambda/(\lambda+\beta)$. We can, therefore, immediately obtain that

$$\mathbb{E}\tilde{N}(k) = \frac{\lambda}{\beta} k, \qquad (26)$$

and

$$\mathbb{V}\mathrm{ar}\tilde{N}(k) = \frac{\lambda(\lambda+\beta)}{\beta^2} k. \qquad (27)$$

Note that the subordinated process $N_1(\tau_k)$, $k \ge 0$, can be interpreted as follows:

$$N_1(\tau_k) \overset{\mathrm{d}}{=} X_1 + \cdots + X_N, \qquad (28)$$

where N is a Poisson random variable with parameter equal to

$$\mu = \log \left(\frac{\lambda+\beta}{\beta}\right)^k, \qquad (29)$$

and the X_js are i.i.d. random variables with logarithmic distribution with parameter $\alpha = \lambda/(\lambda+\beta)$.

Remark 2 (A note on paths)**.** *The subordinated process $\tilde{N}(k)$, has increasing paths with jumps. The jumps are of positive integer-valued size. It can thus possibly used to model branching phenomena in discrete-time, when an arbitrary number of offsprings is permitted.*

4 Poisson and Yule–Furry processes subordinated to the inverse of an independent fractional Poisson process

In this section we introduce the inverse process of a fractional Poisson process independent of $N_1(t)$, $t > 0$. For in-depth information on the latter we refer to [6] or, more recently [3]. In the following, the fractional Poisson process will be indicated as $N^{\nu}(t)$, $t > 0$, where $\nu \in (0, 1]$ is the index of fractionality.

The inverse process is

$$\tau_k^{\nu} = \inf(t \colon N^{\nu}(t) = k), \qquad k \ge 0. \qquad (30)$$

As in the classical case we can write that $\Pr(\tau_k^{\nu} \le t) = \Pr(N^{\nu}(t) \ge k)$. From [7, formula (2.19)], the distribution is

$$\Pr(\tau_k^{\nu} \in \mathrm{d}t)/\mathrm{d}t = \beta^k t^{\nu k - 1} E_{\nu,\nu k}^{k}(-\beta t^{\nu}), \qquad (31)$$

where $\beta > 0$ is the rate of the fractional Poisson process and the function

$$E_{\alpha,\gamma}^{\delta}(z) = \sum_{r=0}^{\infty} \frac{(\delta)_r z^r}{\Gamma(\alpha r + \gamma) r!}, \quad \alpha, \gamma, \delta \in \mathbb{C},\ \Re(\alpha) > 0, \qquad (32)$$

is the so-called generalised Mittag–Leffler function [8, page 91]. The distribution of the subordinated process $\tilde{N}^\nu(k) = N_1(\tau_k^\nu)$ reads

$$\begin{aligned} &\Pr\left(\tilde{N}^\nu(k) = r\right) \quad (33)\\ &= \int_0^\infty \frac{e^{-\lambda s}(\lambda s)^r}{r!} \beta^k s^{\nu k-1} E^k_{\nu,\nu k}(-\beta s^\nu) \mathrm{d}s \\ &= \frac{\lambda^r \beta^k}{r!} \int_0^\infty s^{\nu k+r-1} e^{-\lambda s} E^k_{\nu,\nu k}(-\beta s^\nu) \mathrm{d}s \\ &= \frac{\lambda^r \beta^k}{r!} \frac{\lambda^{-\nu k - r}}{(k-1)!} {}_2\psi_1 \left[\frac{-\beta}{\lambda^\nu} \middle| \begin{array}{l} (k,1),(\nu k + r, \nu) \\ (\nu k, \nu) \end{array} \right], \end{aligned}$$

where the function

$$ {}_p\psi_q \left[z \middle| \begin{array}{l} (a_p, A_p) \\ (b_q, B_q) \end{array} \right] = \sum_{m=0}^\infty \frac{\prod_{j=1}^p \Gamma(a_j + m A_j)}{\prod_{j=1}^q \Gamma(b_j + m B_j)} \frac{z^m}{m!}, \quad (34)$$

is the generalised Wright function. Notice that, in the last step of (33), we used relation (2.3.23) of [8]. The distribution (33) can be further simplified by writing explicitly the Wright function.

$$\begin{aligned} \Pr\left(\tilde{N}^\nu(k) = r\right) &= \frac{1}{r!} \sum_{m=0}^\infty \binom{k+m-1}{m} (-1)^m \quad (35) \\ &\quad \times \left(\frac{\beta}{\lambda^\nu}\right)^{k+m} \frac{\Gamma(\nu(k+m)+r)}{\Gamma(\nu(k+m))}. \end{aligned}$$

From (35), when $\nu = 1$, we retrieve formula (25):

$$\begin{aligned} &\Pr\left(\tilde{N}^1(k) = r\right) \quad (36) \\ &= \sum_{m=0}^\infty (-1)^m \left(\frac{\beta}{\lambda}\right)^{k+m} \frac{(k+m+r-1)!}{r!m!(k-1)!} \\ &= \binom{k+r-1}{r} \sum_{m=0}^\infty \left(\frac{\beta}{\lambda}\right)^{k+m} (-1)^m \binom{k+m+r-1}{m} \\ &= \binom{k+r-1}{r} \left(\frac{\beta}{\lambda}\right)^k \sum_{m=0}^\infty \left(\frac{\beta}{\lambda}\right)^m \binom{-(k+r)}{m} \\ &= \binom{k+r-1}{r} \left(\frac{\beta}{\lambda}\right)^k \left(1+\frac{\beta}{\lambda}\right)^{-(k+r)} \\ &= \binom{k+r-1}{r} \left(\frac{\lambda}{\lambda+\beta}\right)^r \left(1-\frac{\lambda}{\lambda+\beta}\right)^k. \end{aligned}$$

Following the above calculation, it is also possible to determine the distribution of a Yule–Furry process, indicated here as $Y(t)$, $t > 0$, with birth rate $\lambda > 0$, subordinated to τ_k^ν, $k \geq 0$, as follows.

$$\begin{aligned} &\Pr\left(Y(\tau_k^\nu) = r\right) \quad (37) \\ &= \int_0^\infty e^{-\lambda s} \left(1 - e^{-\lambda s}\right)^{r-1} \beta^k s^{\nu k - 1} E^k_{\nu,\nu k}(-\beta s^\nu) \mathrm{d}s \\ &= \beta^k \sum_{j=1}^r \binom{r-1}{j-1} (-1)^{j-1} \\ &\quad \times \int_0^\infty e^{-\lambda j s} s^{\nu k - 1} E^k_{\nu,\nu k}(-\beta s^\nu) \mathrm{d}s \\ &= \beta^k \sum_{j=1}^r \binom{r-1}{j-1} (-1)^{j-1} \frac{1}{\left[(\lambda j)^\nu + \beta\right]^k}. \end{aligned}$$

In the last step we exploited relation (2.3.24) of [8].

5 Poisson process subordinated to the inverse of an independent fractional Yule process

Let $Y^\nu(t)$, $t > 0$, be a fractional Yule process with rate $\beta > 0$ [4]. The probability density function of the kth birth W_k^ν, is [9]:

$$\begin{aligned} &f_{\phi_k^\nu}(t) \quad (38) \\ &= \sum_{m=1}^k \sum_{l=1}^m \binom{m-1}{l-1} (-1)^{l-1} \beta l t^{\nu-1} E_{\nu,\nu}(-\beta l t^\nu), \end{aligned}$$

where $t > 0$ and $\nu \in (0,1]$. It is also the probability density function of the right-inverse process $\phi_k^\nu = \inf(t \colon Y^\nu(t) = k)$, $k \geq 1$. In order to derive the distribution of the subordinated process $N_1(\phi_k^\nu)$, $k \geq 1$, we proceed as in the preceding section.

$$\begin{aligned} &\Pr(N_1(\phi_k^\nu) = r) \quad (39) \\ &= \int_0^\infty \frac{e^{-\lambda s}(\lambda s)^r}{r!} \sum_{m=1}^k \sum_{l=1}^m \binom{m-1}{l-1} \\ &\quad \times (-1)^{l-1} \beta l s^{\nu-1} E_{\nu,\nu}(-\beta l s^\nu) \mathrm{d}s \\ &= \frac{\lambda^r \beta}{r!} \sum_{m=1}^k \sum_{l=1}^m \binom{m-1}{l-1} l (-1)^{l-1} \\ &\quad \times \int_0^\infty e^{-\lambda s} s^{\nu + r - 1} E_{\nu,\nu}(-\beta l s^\nu) \mathrm{d}s \\ &= \frac{1}{r!} \sum_{m=1}^k \sum_{l=1}^m \binom{m-1}{l-1} (-1)^l \\ &\quad \times \sum_{n=0}^\infty \left(-\frac{\beta l}{\lambda^\nu}\right)^{n+1} \frac{\Gamma(\nu(n+1)+r)}{\Gamma(\nu(n+1))}. \end{aligned}$$

References

[1] S. Bochner. *Harmonic Analysis and the Theory of Probability.* University of California Press, Berkeley, 1955.

[2] R. D DeBlassie. Iterated Brownian motion in a an open set. *Ann. Appl. Probab.*, 14(3):1529–1558, 2004.

[3] L. Beghin and E. Orsingher. Fractional Poisson processes and related planar random motions. *Electron. J. Probab.*, 14(61):1790–1826, 2009.

[4] E. Orsingher and F. Polito. Fractional pure birth processes. *Bernoulli*, 16(3):858–881, 2010.

[5] K. N. Boyadzhiev. Exponential Polynomials, Stirling Numbers, and Evaluation of Some Gamma Integrals. *Abstr. Appl. Anal.*, 2009.

[6] N. Laskin. Fractional Poisson process. *Commun. Nonlinear Sci. Numer. Simul.*, 8(3–4):201–210, 2003.

[7] L. Beghin and E. Orsingher. Poisson-type processes governed by fractional and higher-order recursive differential equations. *Electron. J. Probab.*, 15(22):684–709, 2010.

[8] A. M. Mathai and H. J. Haubold. *Special Functions for Applied Scientists*. Springer, New York, 2008.

[9] D. O. Cahoy and F. Polito. Simulation and Estimation for the Fractional Yule Process. *Submitted*, 2010.